Copulas as an Integrated Risk Management Tool

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Part 1

Copula Ideas

I Dependence concepts

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Drawbacks of linear correlation

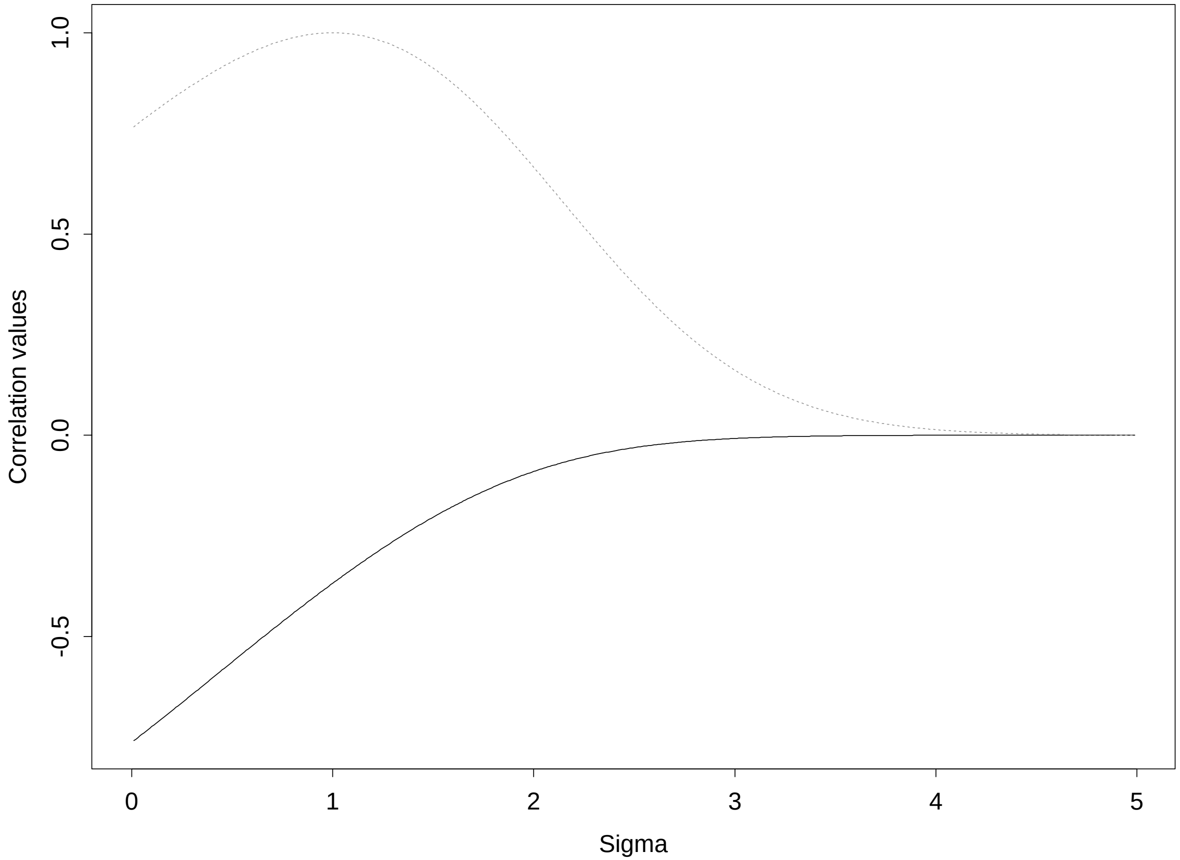
- Linear correlation of random variables X, Y is not defined if the variance of X or Y is infinite.

- Linear correlation can easily be misinterpreted.

- Linear correlation is *not* invariant under non-linear strictly increasing transformations $T : \mathbb{R} \longrightarrow \mathbb{R}$, i.e.

$$\rho\,(T(X), T(Y)) \neq \rho\,(X, Y)\,.$$

- Given distribution functions F and G for X and Y, in general not all linear correlations between -1 and 1 can be obtained by a suitable choice of the joint distribution.



Upper and lower bounds for $\rho(X,Y)$, where $X \sim$ Lognormal$(0,1)$ and $Y \sim$ Lognormal$(0,\sigma^2)$.

Note:
For $\sigma = 4$, $\rho(X,Y) = 0.01372$ in fact means that $Y = T(X)$, with T strictly increasing!

Copulas

Definition

A copula, $C : [0,1]^n \mapsto [0,1]$, is a joint distribution function (d.f.) of n random variables uniformly distributed on $[0,1]$.

- Let H be an n-dimensional d.f. with continuous margins $F_1, \ldots, F_n$. Then

  $$H(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n)).$$

  Conversely, if C is a copula and $F_1, \ldots, F_n$ are d.f.s, then H is an n-dimensional d.f. with margins $F_1, \ldots, F_n$.

  Hence the copula of $(X_1, \ldots, X_n) \sim H$ is the d.f. of $(F_1(X_1), \ldots, F_n(X_n))$.

- If $F_1, \ldots, F_n$ are strictly increasing d.f.s, then for every $\mathbf{u} = (u_1, \ldots, u_n)$ in $[0,1]^n$,

  $$C(\mathbf{u}) = H(F_1^{-1}(u_1), \ldots, F_n^{-1}(u_n)).$$

Special copulas

$$
\begin{aligned}
M^n(\mathbf{u}) &= \min(u_1, u_2, \ldots, u_n) \\
W^n(\mathbf{u}) &= \max(u_1 + u_2 + \cdots + u_n - n + 1, 0) \\
\Pi^n(\mathbf{u}) &= u_1 u_2 \ldots u_n
\end{aligned}
$$

Note: M^n and Π^n are copulas for all $n \geq 2$ but W^n is a copula only for $n = 2$.

Definition

1. $X_1, \ldots, X_n$ comonotonic $\iff$ $(X_1, \ldots, X_n)$ has copula M^n
   $\iff$ $(X_1, \ldots, X_n) =_d (\alpha_1(Z), \ldots, \alpha_n(Z))$, $\alpha_1, \ldots, \alpha_n$ increasing and Z is some real valued random variable.

2. X, Y countermonotonic $\iff$ (X, Y) has copula W^2
   $\iff$ $(X, Y) =_d (\alpha(Z), \beta(Z))$, α inc., β dec. and Z is some real valued r.v.

3. $X_1, \ldots, X_n$ independent $\iff$ $(X_1, \ldots, X_n)$ has copula Π^n.

Properties of copulas

Bounds

For every $\mathbf{u} \in [0,1]^n$ we have

$$W^n(\mathbf{u}) \le C(\mathbf{u}) \le M^n(\mathbf{u}).$$

These bounds are the best possible.

Strictly increasing transformations

If $\alpha_1, \ldots, \alpha_n$ are strictly increasing functions, then $(\alpha_1(X_1), \ldots, \alpha_n(X_n))$ has the same copula as $(X_1, \ldots, X_n)$.

Consequence:

Dependence measures expressed only in terms of the copula are invariant under strictly increasing transformations of the underlying random variables.

Rank correlations

Let (X, Y) be a random vector with continuous margins F and G and copula C.

- *Kendall's tau* of (X, Y) is given by

$$\begin{aligned}\tau(X,Y) &= \mathbb{P}[(X-X')(Y-Y')>0]-\mathbb{P}[(X-X')(Y-Y')<0] \\ &= 4\iint_{[0,1]^2} C(u,v)\,\mathrm{d}C(u,v)-1,\end{aligned}$$

  where (X, Y) and (X', Y') are independent copies.

- *Spearman's rho* of (X, Y) is given by

$$\begin{aligned}\rho_S(X,Y) &= \rho(F(X),G(Y)) \\ &= 12\iint_{[0,1]^2} uv\,\mathrm{d}C(u,v)-3.\end{aligned}$$

Kendall's tau and Spearman's rho are called rank correlations.

Properties of rank correlation

Let X and Y be continuous random variables with copula C, and let δ denote Kendall's tau or Spearman's rho. The following properties are not shared by linear correlation.

- If T is strictly monotone, then

$$\delta(T(X),Y) = \delta(X,Y), \; T \text{ increasing},$$
$$\delta(T(X),Y) = -\delta(X,Y), \; T \text{ decreasing}.$$

- $\delta(X,Y) = 1 \quad \iff \quad C = M^2$

- $\delta(X,Y) = -1 \quad \iff \quad C = W^2$

- $\delta(X,Y)$ depends only on the copula of (X,Y).

Given a rank correlation matrix there is always a multivariate distribution with this rank correlation matrix, regardless of the choice of margins. This is *not* true for linear correlation.

Tail dependence

Let (X, Y) be a random vector with continuous margins F and G and copula C.

- The coefficient of upper tail dependence of (X, Y) is

$$\begin{aligned} \lambda_U \quad &:= \lim_{u \nearrow 1} \mathbb{P}[Y > G^{-1}(u) | X > F^{-1}(u)] \\ &= \lim_{u \nearrow 1} (1 - 2u + C(u, u))/(1 - u) \end{aligned}$$

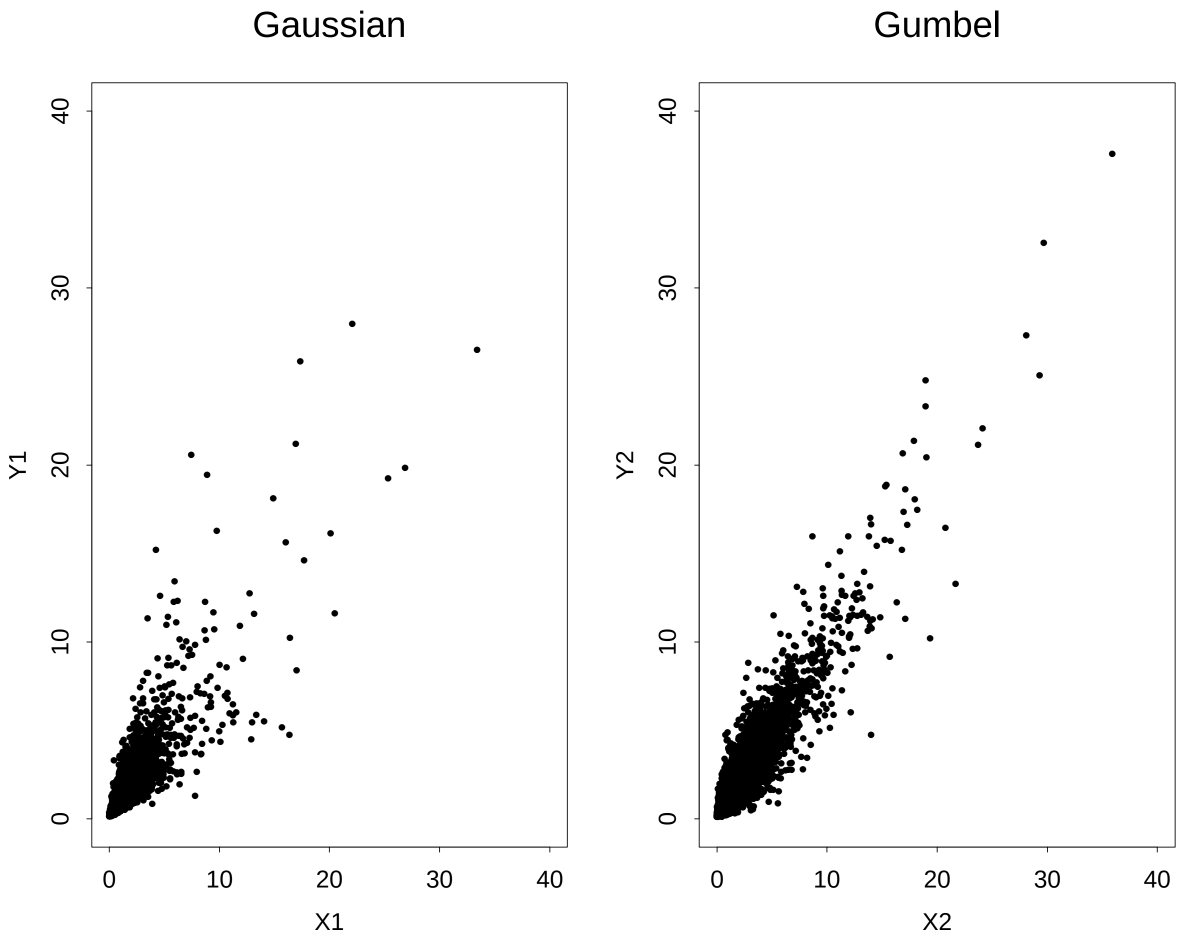
  provided that the limit $\lambda_U \in [0, 1]$ exists.
  If $\lambda_U > 0$, then (X, Y) has upper tail dependence.

- If

$$\lim_{u \searrow 0} C(u, u)/u = \lambda_L > 0$$

  exists, then (X, Y) has lower tail dependence.

- Tail dependence is a *copula* property.



Two bivariate distributions with standard lognormal margins and Kendall's tau 0.7, but different dependence structures.

Gumbel copulas have upper tail dependence, but *Gaussian* copulas have not.

Elliptical copulas

A *spherical distribution* is an extension of the multivariate normal distribution $\mathcal{N}_n(\mathbf{0}, \mathbf{I}_n)$ and an *elliptical distribution* is an extension of $\mathcal{N}_n(\mu, \mathbf{\Sigma})$.

Recall that $\mathcal{N}_n(\mu, \mathbf{\Sigma})$ can be defined as the distribution of

$$\mathbf{X} = \mu + A\mathbf{Y},$$

where $\mathbf{Y} \sim \mathcal{N}_n(\mathbf{0}, \mathbf{I}_n)$ and $\mathbf{\Sigma} = AA^T$.

$\mathbf{X}$ has a spherical distribution if

$$\mathbf{X} =_d R\mathbf{U}$$

for some random variable $R \geq 0$ independent of the random vector $\mathbf{U}$ uniformly distributed on the unit hypersphere.

$\mathbf{X}$ has an elliptical distribution with parameters μ and $\mathbf{\Sigma}$ if

$$\mathbf{X} =_d \mu + A\mathbf{Y},$$

where $\mathbf{Y}$ is spherical and $\mathbf{\Sigma} = AA^T$.

Gaussian copulas

Let $\mathbf{Z} = (Z_1, \ldots, Z_n) \sim \mathcal{N}_n(\mu, \mathbf{\Sigma})$ with non-degenerate margins and let F_i be the d.f. of $Z_i \sim \mathcal{N}(\mu_i, \Sigma_{ii})$. Then

$$(F_1(Z_1), \ldots, F_n(Z_n))$$

has d.f.

$$C_\rho^{\mathsf{Ga}}(\mathbf{u}) = \Phi_\rho^n(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_n)),$$

where ρ is the linear correlation matrix corresponding to the covariance matrix $\mathbf{\Sigma}$.

If $G_1, \ldots, G_n$ are strictly increasing d.f.s, then

$$(G_1^{-1}(F_1(Z_1)), \ldots, G_n^{-1}(F_n(Z_n)))$$

has copula C_ρ^{Ga} and margins $G_1, \ldots, G_n$.

Note that ρ is not the correlation matrix of the above vector.

t-copulas

Let $\mathbf{Z} = (Z_1, \ldots, Z_n) \sim \mathcal{N}_n(\mathbf{0}, \mathbf{\Sigma})$ with non-degenerate margins and let

$$\mathbf{X} = \mu + \frac{\sqrt{\nu}}{\sqrt{S}}\mathbf{Z},$$

where $\mathbf{Z}$ and $S \sim \chi^2_\nu$ are independent.

$\mathbf{X}$ has a t-distribution with ν degrees of freedom, mean μ (if $\nu > 1$) and covariance matrix $\frac{\nu}{\nu-2}\Sigma$ (if $\nu > 2$).

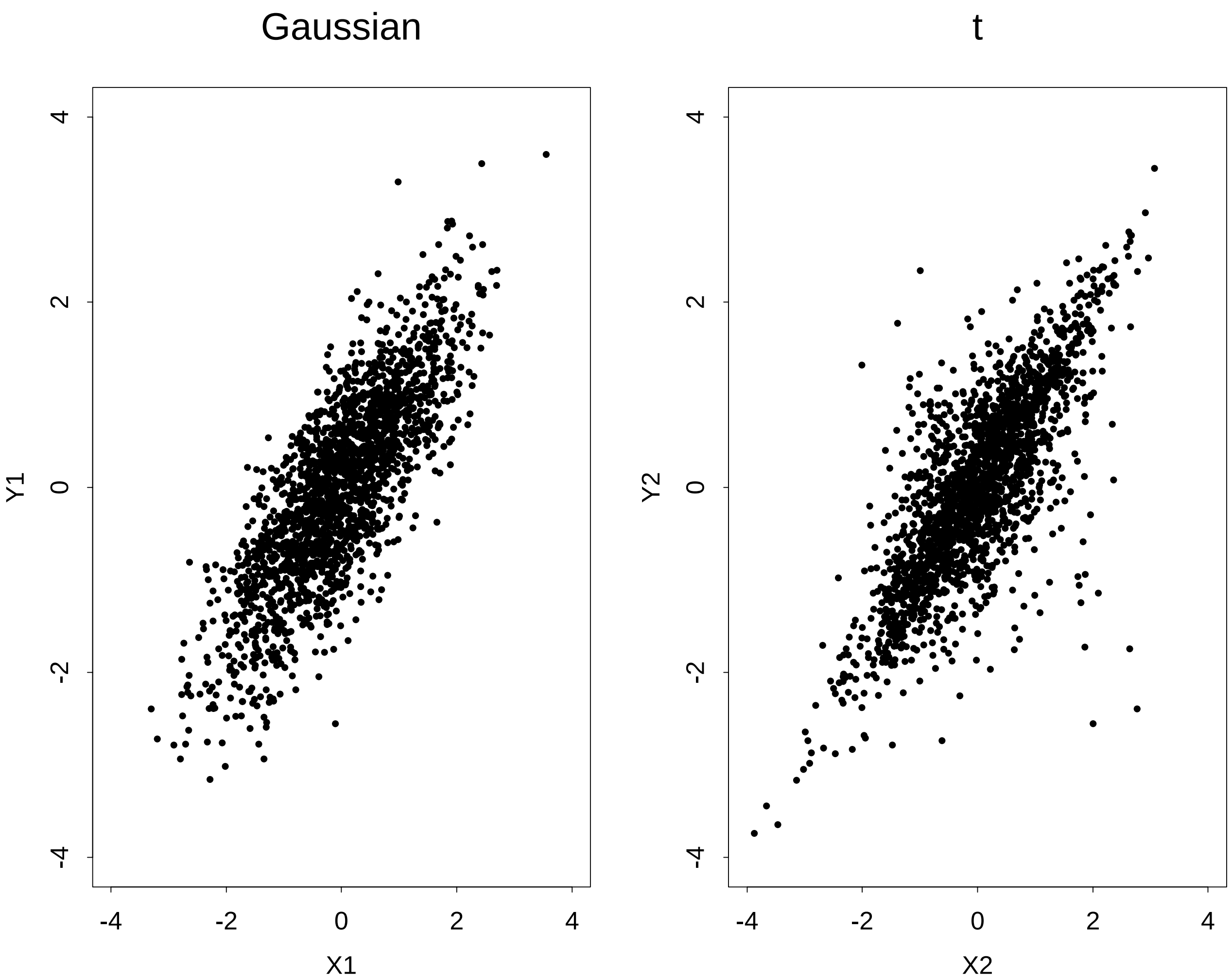
If X_i has d.f. G_i, then the d.f. of

$$(G_1(X_1), \ldots, G_n(X_n))$$

is the t_ν-copula $C^{\mathrm{t}}_{\nu,\rho}$, where ρ is the linear correlation matrix corresponding to $\mathbf{\Sigma}$. $C^{\mathrm{t}}_{\nu,\rho}$ has upper and equal lower tail dependence.

For all elliptical copulas

$$\tau(i,j) = \frac{2}{\pi}\arcsin \rho(i,j).$$



Two bivariate distributions with standard normal margins and $\tau = 0.6$. Gaussian and t_2-copulas.

Archimedean copulas

Let φ be a continuous strictly decreasing convex function from $[0,1]$ to $[0,\infty]$ such that $\varphi(1)=0$. Then

$$\varphi^{[-1]}(\varphi(u)+\varphi(v)), \qquad u,v\in[0,1],$$

is a copula with generator φ, where

$$\varphi^{[-1]}(t)=\begin{cases}\varphi^{-1}(t), & t\le\varphi(0),\\ 0, & t\ge\varphi(0).\end{cases}$$

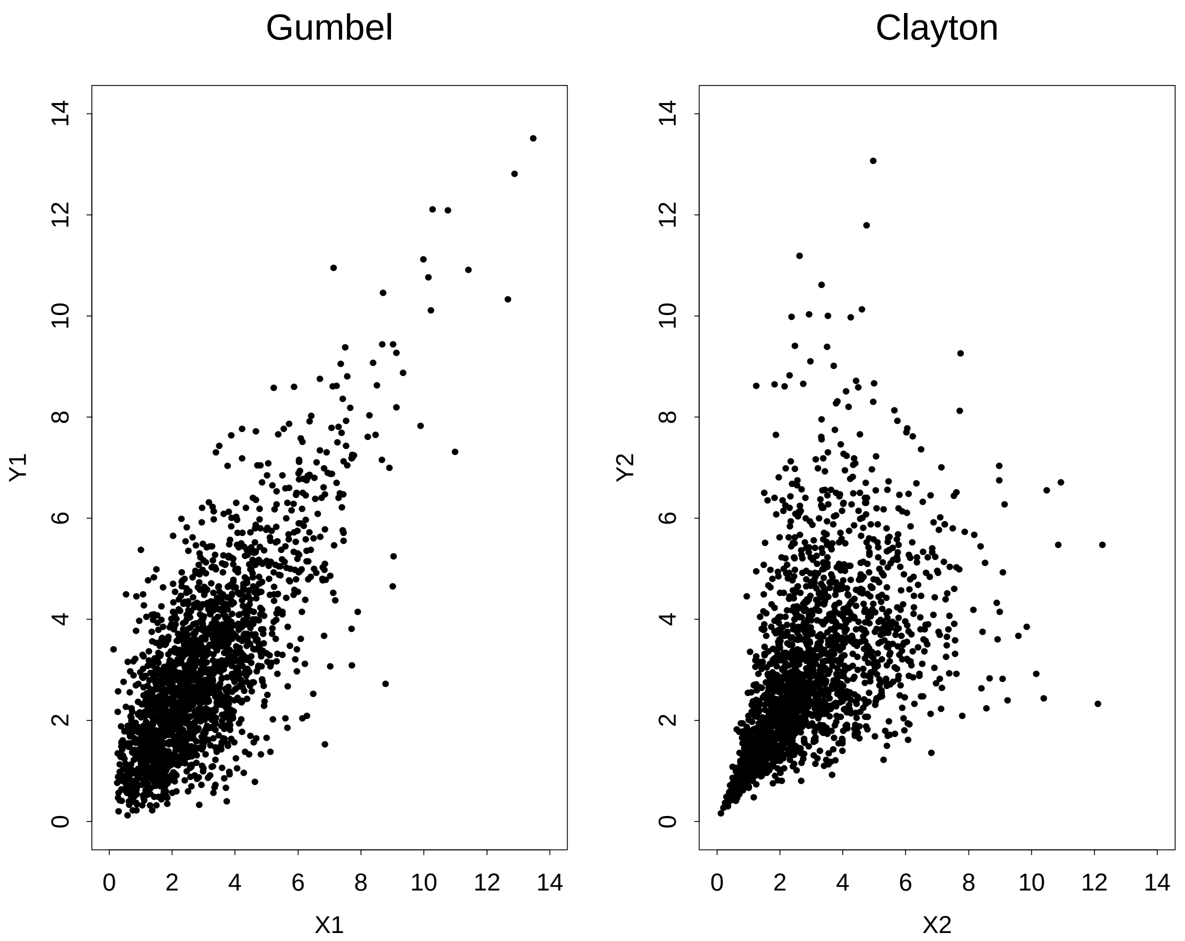
If $\varphi(0)=\infty$, then $\varphi^{[-1]}=\varphi^{-1}$.

- Gumbel copula: Take $\varphi(t)=(-\ln t)^{\theta}$ with $\theta\in[1,\infty)$,

$$C_{\theta}(u,v)=\exp(-[(-\ln u)^{\theta}+(-\ln v)^{\theta}]^{1/\theta}).$$

- Clayton copula: Take $\varphi(t)=(t^{-\theta}-1)/\theta$ with $\theta\in[-1,\infty)\backslash\{0\}$,

$$C_{\theta}(u,v)=\max([u^{-\theta}+v^{-\theta}-1]^{-1/\theta},0).$$



Two bivariate distributions with Gamma$(3, 1)$ margins and $\tau = 0.5$.

Gumbel copulas have upper tail dependence, Clayton copulas have lower tail dependence.

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Credit risk example

- Risks $X_1, \ldots, X_n$.
- Thresholds $k_1, \ldots, k_n$.
- Default occurs for company i if $X_i > k_i$.

What is the probability of at least l defaults?

Assume historical data are available allowing estimation of
- marginal distributions,
- pairwise *rank* correlations.

$N = \left|\left\{i \in \{1, \ldots, n\} \,\middle|\, X_i > k_i\right\}\right|$ is the number of defaults.

The probability of all companies defaulting is given by

$$\begin{aligned}\mathbb{P}(N = n) &= \overline{H}(k_1, \dots, k_n) \\ &= \widehat{C}(\overline{F}_1(k_1), \dots, \overline{F}_n(k_n)),\end{aligned}$$

where $\widehat{C}$ is the *survival copula* of $(X_1, \dots, X_n)$.
We can evaluate $\mathbb{P}(N \geq l)$ for various copulas.

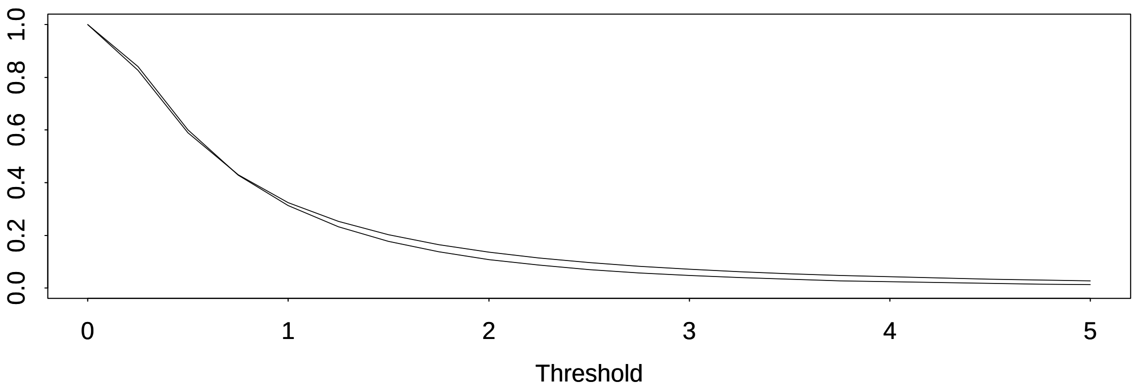
Illustration: $l = n = 3$; $X_i \sim$ Lognormal$(0, 1)$; $k_i = k$ for all i;
$\tau(X_i, X_j) = 0.5$ for all $i \neq j$.

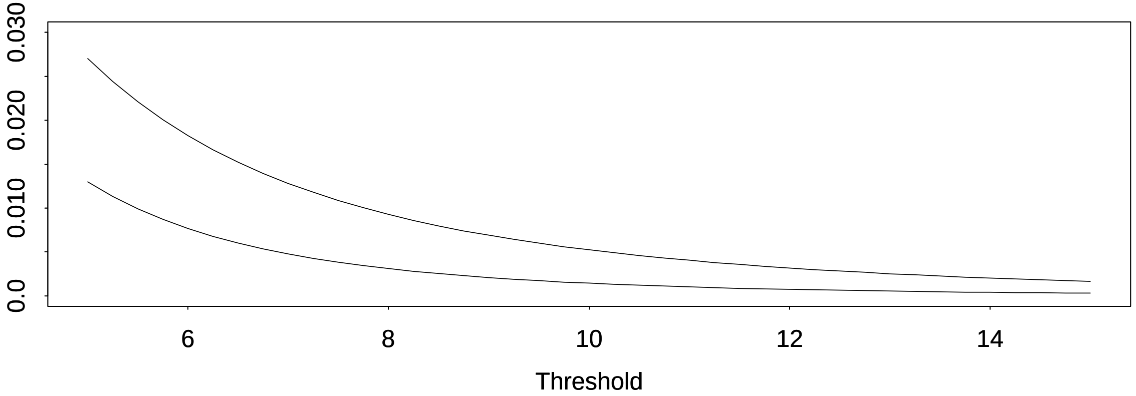
We compare trivariate Gaussian and Gumbel copulas and use the relations

$$\rho = \sin(\pi\tau/2) \quad \text{and} \quad \theta = \frac{1}{1 - \tau}$$

to parametrisize the respective copulas so that they have a common Kendall's tau rank correlation matrix.

Probability of joint default





Note that for $k = 5 \approx \text{VaR}_{0.95}(X_i)$ $\dfrac{\mathbb{P}^{\text{Gumbel}}\{N=3\}}{\mathbb{P}^{\text{Gaussian}}\{N=3\}} \approx 2$

and that for $k = 10 \approx \text{VaR}_{0.99}(X_i)$ $\dfrac{\mathbb{P}^{\text{Gumbel}}\{N=3\}}{\mathbb{P}^{\text{Gaussian}}\{N=3\}} \approx 4.$

Market risk example

Linear portfolio of p equities with value at time t given by

$$V_t = \sum_{i=1}^{p} \beta_i S_{i,t},$$

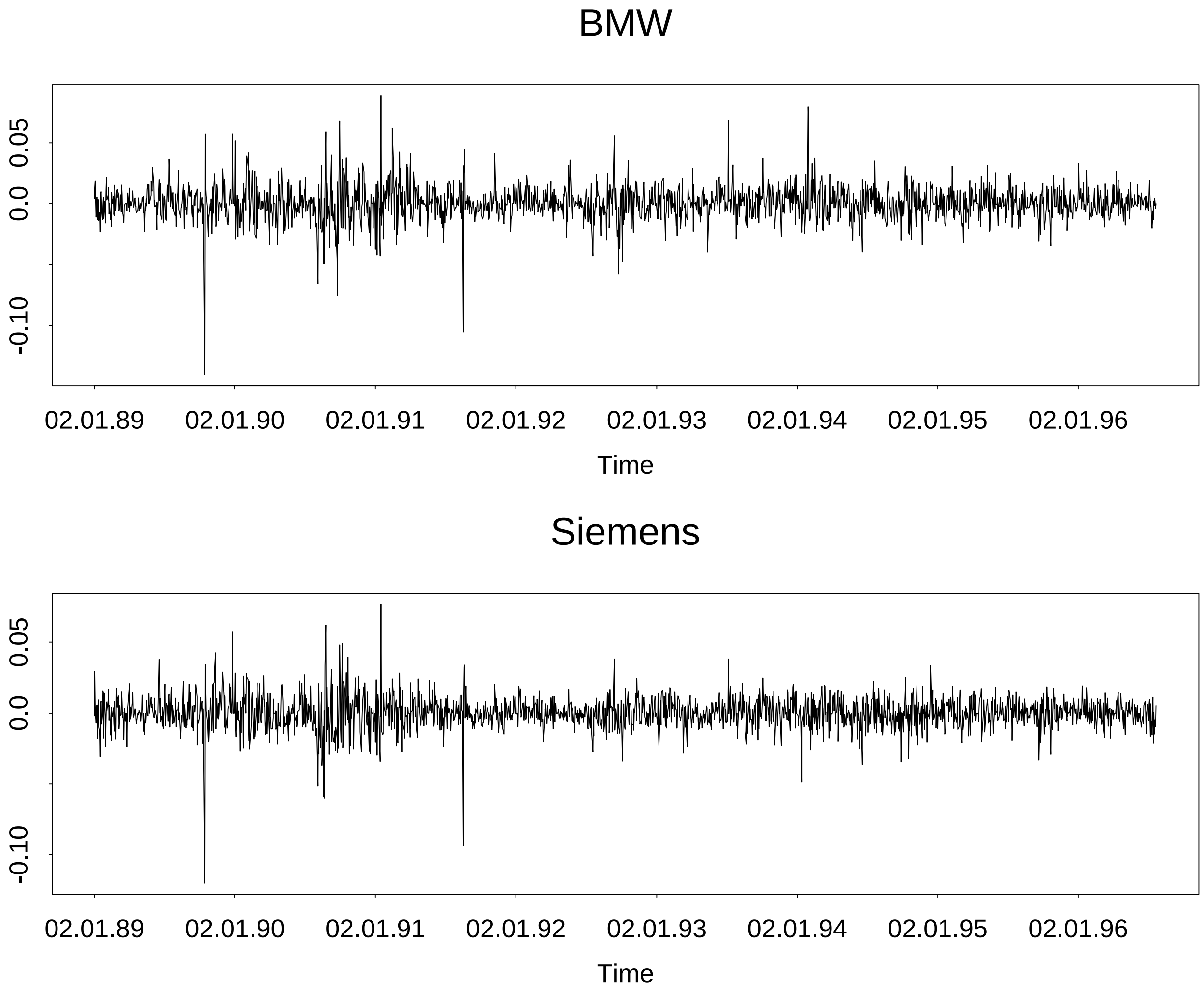
β_i : # units of equity i, $S_{i,t}$: price of equity i.

Let $\Delta_{t+1} = -(V_{t+1} - V_t)/V_t$, the relative loss over time period $(t, t+1]$, be our aggregate risk. Then

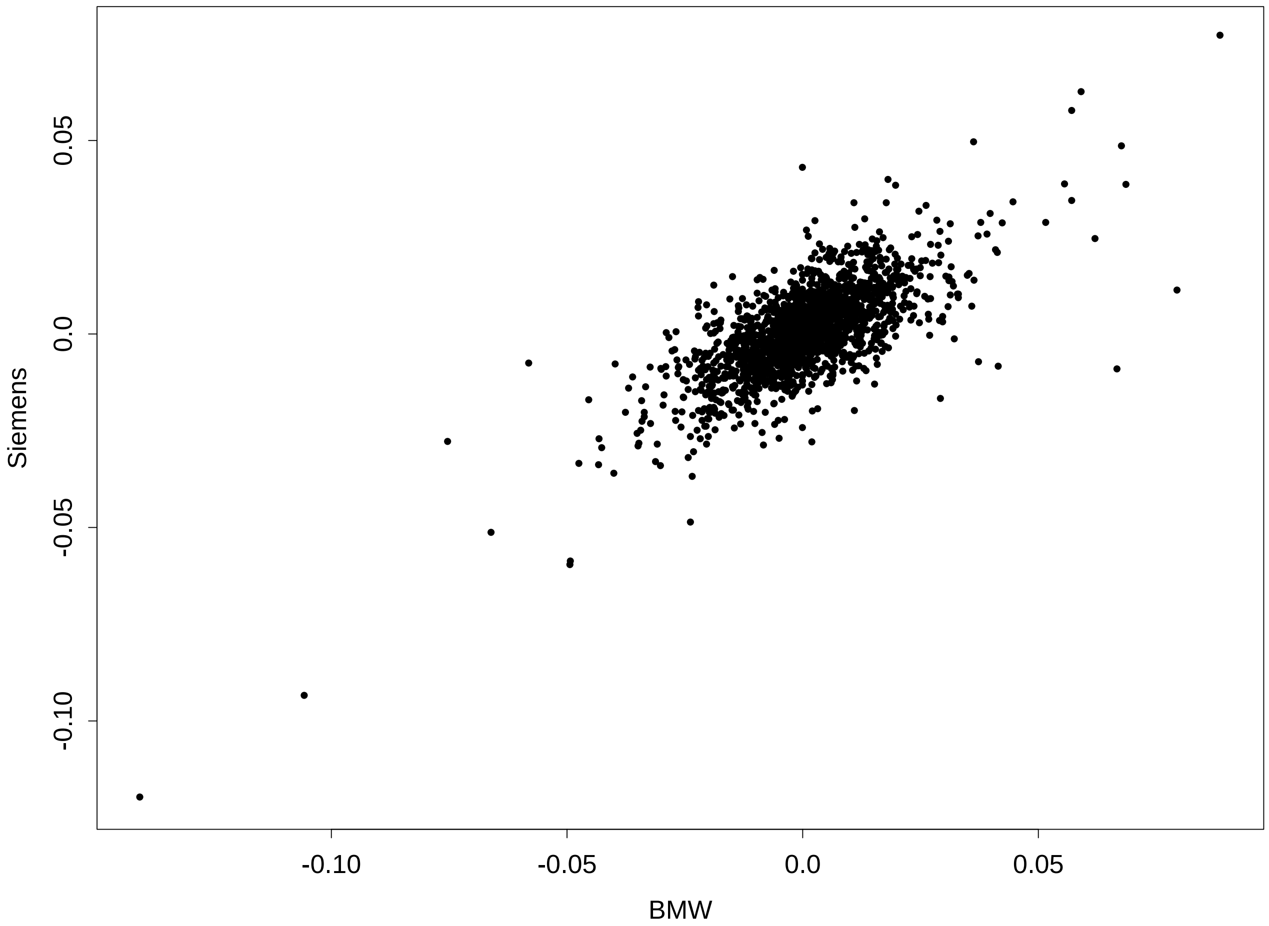
$$\Delta_{t+1} = \sum_{i=1}^{p} \gamma_{i,t}\delta_{i,t+1},$$

$\gamma_{i,t} = \beta_i S_{i,t}/V_t$ and $\delta_{i,t+1} = -(S_{i,t+1} - S_{i,t})/S_{i,t}$.

We stand at time t. Set $\gamma := (\gamma_{1,t}, \ldots, \gamma_{p,t})^T$, $\delta := (\delta_{1,t+1}, \ldots, \delta_{p,t+1})^T$ and $\Delta := \Delta_{t+1}$.



BMW and Siemens daily log returns.



BMW and Siemens daily log returns.

A naive approach

Suppose $\delta \sim \mathcal{N}_p(\mu, \Sigma)$.

$$\Rightarrow \Delta \sim \mathcal{N}(\gamma^T \mu, \gamma^T \Sigma \gamma)$$

During periods of high volatility *sample* correlations are higher. Suppose that conditioned on being in such a period $\delta^* \sim \mathcal{N}_p(\mu^*, \Sigma^*)$.

$$\Rightarrow \Delta^* \sim \mathcal{N}(\gamma^T \mu^*, \gamma^T \Sigma^* \gamma)$$

Consequence: For $\alpha \in (0.5, 1)$

$$\frac{VaR_\alpha(\Delta^*) - \gamma^T \mu^*}{VaR_\alpha(\Delta) - \gamma^T \mu} = \sqrt{\frac{\gamma^T \Sigma^* \gamma}{\gamma^T \Sigma \gamma}}$$

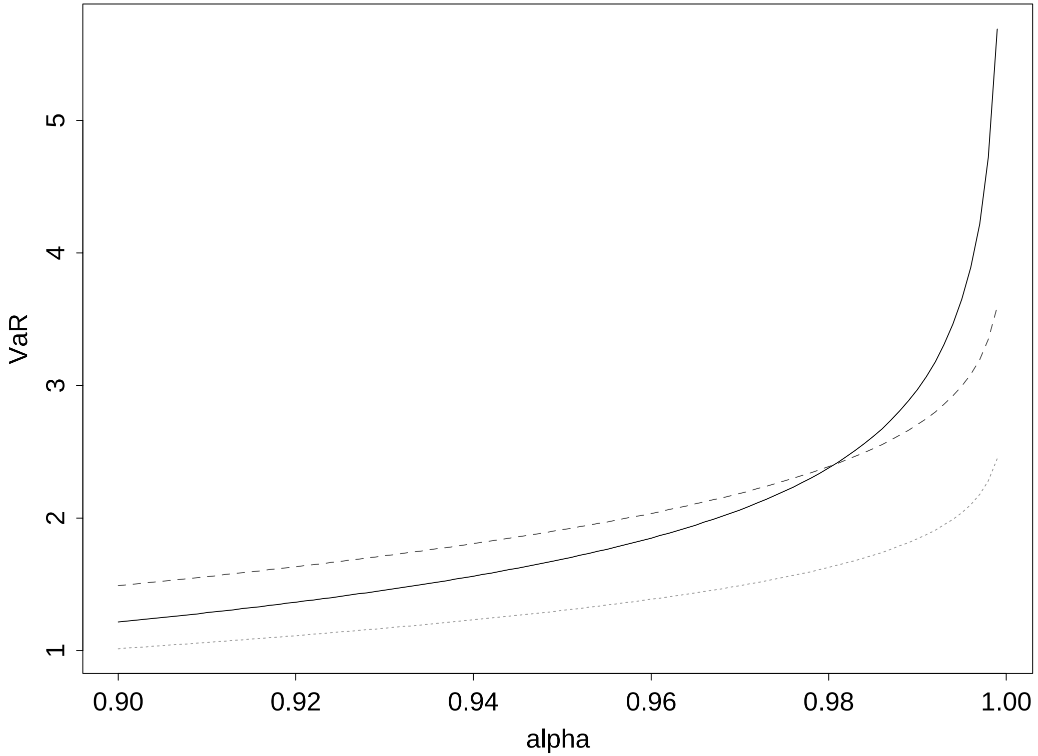
Problems:

- No evidence that correlation really changes,
- Margins heavier tailed than Normal,
- Equity data suggest tail dependence.

Illustration

- $p = 10$, $\gamma^T = (1, \ldots, 1)/10$,
- $\tau_{ij} = 0.4$, $\tau^*_{ij} = 0.6$ for all $i \neq j$,
- $\mu_i = 0$, $\Sigma_{ij} = \sin(\pi\tau_{ij}/2)$,
- $\mu^*_i = 0$, $\Sigma^*_{ij} = 1.5\sin(\pi\tau^*_{ij}/2)$,
- $\delta \sim \mathcal{N}_p(\mu, \Sigma)$, $\delta^* \sim \mathcal{N}_p(\mu^*, \Sigma^*)$,
- $\tilde{\delta} \sim t_{\nu,p}(\mu, \Sigma)$ with $\nu = 4$.

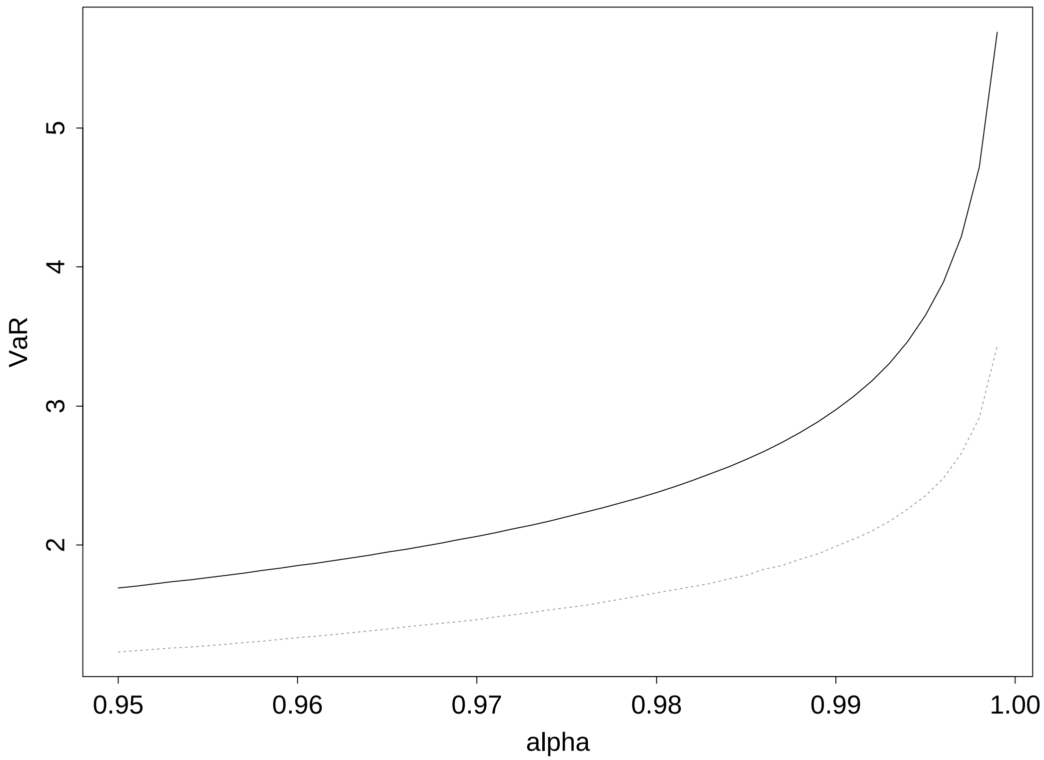
VaR comparison



The lower dashed curve shows $\text{VaR}_\alpha(\Delta)$, the upper dashed curve $\text{VaR}_\alpha(\Delta^*)$ and the solid curve $\text{VaR}_\alpha(\tilde{\Delta})$ for $\alpha \in [0.9, 0.999]$.

Is the difference due to the heavier tailed t_4-margins of $\tilde{\delta}$?

VaR comparison



The upper curve shows $\text{VaR}_\alpha(\tilde{\Delta})$. The lower curve shows $\text{VaR}_\alpha(\Delta')$, where δ' has a Gaussian copula, t_4-margins and the same covariance matrix and mean vector as $\tilde{\delta}$.

Heavy tailed margins do not explain the difference!