Different Kinds of Risk

Paul Embrechts[1], Hansjörg Furrer[2], and Roger Kaufmann[3]

[1] ETH Zurich, Department of Mathematics, 8092 Zurich, Switzerland
embrechts@math.ethz.ch
[2] Swiss Life, General-Guisan-Quai 40, 8022 Zurich, Switzerland
Hansjoerg.Furrer@swisslife.ch
[3] Swiss Life, General-Guisan-Quai 40, 8022 Zurich, Switzerland
Roger.Kaufmann@swisslife.ch

Summary. Over the last twenty years, the financial industry has developed numerous tools for the quantitative measurement of risk. The need for this was mainly due to changing market conditions and regulatory guidelines. In this article we review these processes and summarize the most important risk categories considered.

1 Introduction

Tumbling equity markets, falling real interest rates, an unprecedented increase in longevity, inappropriate reserving, and wrong management decisions were among the driving forces that put the financial stability of so many (insurance) companies at risk over the recent past. Senior management, risk managers, actuaries, accounting conventions, regulatory authorities all played their part. With the solvency of many companies put at stake, political intervention led to the revision of the existing regulatory frameworks. For both the insurance and the banking industry, the aim is to create prudential supervisory frameworks that focus on the true risks being taken by a company.

In the banking regime, these principles were set out by the Basel Committee on Banking Supervision (the "Committee") and culminated in the so-called Basel II Accord, see [BII]. Initially and under the original 1988 Basel I Accord, the focus has been on techniques to manage and measure market and credit risk. *Market risk* is the risk that the value of the investments will change due to moves in the market risk factors. Typical market risk factors are stock prices or real estate indices, interest rates, foreign exchange rates, commodity prices. *Credit risk*, in essence, is the risk of loss due to counter-party defaulting on a contract. Typically, this applies to bonds where the bond holders are concerned that the counter-party may default on the payments (coupon or principal). The goal of the new Basel II Accord was to overturn the imbalances that prevailed in the original 1988 accord. Concomitant with the arrival of Basel II and its more risk sensitive capital requirements for market and credit risk, the Committee introduced a new risk category aiming at capturing risks "other than market and credit risks". The introduction of the *operational risk* category was motivated, among other considerations, by events such as the Barings Bank failure. The Basel Committee defines operational risk as the risk of loss resulting from inadequate or failed internal processes, people and systems, or from external events. The Basel II definition includes legal risk, but excludes strategic risk, i.e. the risk of a loss arising from a poor strategic business decision. Furthermore, this definition excludes reputational risk. Examples of operational risk include, among others, technology failure, business premises becoming unavailable, errors in data processing, fraud, etc. The capital requirement of Basel II is that banks must hold capital of at least 8% of total risk-weighted assets. This definition was retained from the original accord.

Insurance regulation too is rapidly moving towards risk-based foundations. Based on the findings of the Müller Report [Mul97], it was recognized that a fundamental review of the assessment of the overall financial position of an insurance company should be done, including for example the interactions between assets and liabilities, accounting systems and the methods to calculate the solvency margins. In 2001, the European Commission launched the so-called Solvency II project. The key objective of Solvency II is to secure the benefits of the policyholders thereby assessing the company's overall risk profile. A prudential supervisory scheme does not strive for a "zero-failure" target; in a free market, failures will occur. Rather, prudential supervisory frameworks should be designed in such a way that a smooth run-off of the portfolios is ensured in case of financial distress. Phase 1 of the Solvency II project began in 2001 with the constitution of the so-called London Working Group chaired by Paul Sharma from the FSA (Financial Services Authority). The resulting Sharma Report [Sha02] was published in 2002, and contains a survey of actual failures and near misses from 1996 to 2001. The second phase lasts from 2003 to 2007 and is designated to the development of more detailed rules. Finally, the third phase should be terminated by 2010, and is devoted to the implementation of the new standards, also in the national laws.

At the heart of both Basel II and Solvency II lies a *three pillar structure*. Pillar one defines the minimum financial requirements. The second pillar earmarks the supervisory review process, whereas pillar three sets out the disclosure requirements. The minimum financial requirements relate a company's available capital to its economic capital. Economic capital is the amount of capital that is needed to support for retained risks in a loss situation. Associating available capital with economic capital is only meaningful if consistency prevails between valuation and risk measurement. The arrival of a robust marked-to-market culture in the 1990s helps to achieve greater harmonization in this context.

The three-pillar structure of both risk based insurance and banking supervisory frameworks indicates that the overall assessment of a financial institution's financial stability goes beyond the determination of capital adequacy ratios. Nevertheless, the focus in this note will be on the capital requirements, that is on pillar one. More specifically, we address the issue of how to quantify market, credit and insurance risk. We also touch upon the measurement of operational risk. But rather than promoting seemingly sophisticated (actuarial) measurement techniques for quantifying operational risk, we focus on the very special nature of this risk category, implying that standard analytical concepts prove insufficient and also yield counter-intuitive results in terms of diversification.

In hindsight, the inexperienced reader could be tempted to believe that only regulators demand for distinctive risk management cultures and cutting-edge economic capital models. Alas, there are many more institutions that keep a beady eye on the companies' risk management departments: analysts, investors, and rating agencies, to name a few, have a growing interest in what is going on on the risk management side. Standard and Poor's for instance, a rating agency, recently added an "Enterprise Risk Management" (ERM) criterion when rating insurance companies. The ERM rating is based on five key metrics, among which are the risk and economic capital models of insurance undertakings.

The remainder of this note is organized as follows. In Section 2 we provide the basic prerequisites for quantitative risk management by introducing the notion of risk measures and the concept of risk factor mapping. Special emphasis will be given to two widely used risk measures, namely Value at Risk (VaR) and expected shortfall. Section 3 is devoted to the measurement of credit risk, whereas Section 4 deals with market risk. The problem of how to scale a short term VaR to a longer term VaR will be addressed in Section 4.3. The particularities of operational risk loss data and their implications on the economic capital modeling in connection with VaR will be discussed in Section 5. Section 6 is devoted to the measurement of insurance risk. Both the life and non-life measurement approach that will be presented originate from the Swiss Solvency Test. In Section 7 we make some general comments on the aggregation of risks in the realm of economic capital modeling. Attention will be drawn to risks that exhibit special properties such as extreme heavy-tailedness, extreme skewness, or a particular dependence structure.

2 Preliminaries

Risk models typically aim at quantifying likely losses of a portfolio over a given time horizon that could incur for a variety of risks. Formal risk modeling for instance is required under the new (risk-sensitive) supervisory frameworks in the banking and insurance world (Basel II, Solvency II). In this section, we provide the prerequisites for the modeling of risk by introducing risk measures and the notion of risk factor mapping.

2.1 Risk measures

The central notion in actuarial and financial mathematics is the notion of uncertainty or risk. In this article, uncertainty or risk will always be represented by a random variable, say X or $X(t)$, defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T^*]}, \mathbb{P})$. The filtration $(\mathcal{F}_t)_t$ is assumed to satisfy the "usual conditions", that is a) $(\mathcal{F}_t)_t$ is right-continuous and b) $\mathcal{F}_0$ contains all null sets, i.e. if $B \subset A \in \mathcal{F}_0$ with $\mathbb{P}[A] = 0$, then $B \in \mathcal{F}_0$.

Since risks are modeled as (non-negative) random variables, measuring risk is equivalent to establishing a relation ϱ between the set of random variables and $\mathbb{R}$, the real numbers. Put another way, a risk measure is a function mapping a risk X to a real number $\varrho(X)$. If for example X defines the loss in a financial portfolio over some time horizon, then $\varrho(X)$ can be interpreted as the additional amount of capital that should be set aside so that the portfolio becomes acceptable for a regulator, say. The definition of a risk measure is very general, and yet risk measures should fulfill certain properties to make them "good" risk measures. For instance, it should always hold that $\varrho(X)$ is bounded by the largest possible loss, as modeled by F_X. Within finance, Artzner et al. [ADEH99] pioneered the systematic study of risk measure properties, and defined the class of so-called "coherent risk measures" to be the ones satisfying the following properties:

(a) Translation invariance: $\varrho(X + c) = c + \varrho(X)$, for each risk X and constant $c > 0$.
(b) Positive homogeneity: $\varrho(cX) = c\varrho(X)$, for all risks X and constants $c > 0$.
(c) Monotonicity: if $X \leq Y$ a.s., then $\varrho(X) \leq \varrho(Y)$.
(d) Subadditivity: $\varrho(X + Y) \leq \varrho(X) + \varrho(Y)$.

Subadditivity can be interpreted in the way that "a merger should not create extra risk"; it reflects the idea that risk in general can be reduced via diversification.

Note that there exist several variations of these axioms depending on whether or not losses correspond to positive or negative values, or whether a discount rate over the holding period is taken into account. In our case, we consider losses as positive and neglect interest payments. The random variables X, Y correspond to values of risky positions at the end of the holding period, hence the randomness.

Value at Risk

The most prominent risk measure undoubtedly is *Value at Risk* (VaR). It refers to the question of how much a portfolio position can fall in value over a certain time period with a given probability. The concept of Value at Risk originates from J.P. Morgan's RiskMetrics published in 1993. Today, VaR is the key concept in the banking industry for determining market risk capital charges. A textbook treatment of VaR and its properties is Jorion [Jor00]. Formally, VaR is defined as follows:

Definition 1. *Given a risk X with cumulative distribution function F_X and a probability level $\alpha \in (0,1)$, then*

$$\mathrm{VaR}_\alpha(X) = F_X^{-1}(\alpha) = \inf\{x \in \mathbb{R} : F_X(x) \geq \alpha\}.$$

Typical values for α are 0.95, 0.99 or 0.999. The Basel II approach for a market risk charge for example requires a holding period of ten days and a confidence level of $\alpha = 0.99$. At the trading floor level, individual trading limits are typically set for one day, $\alpha = 0.95$.

Even though VaR has become the benchmark risk measure in the financial world, it has some deficiencies which we shall address here. First, note that VaR only considers the result at the end of the holding period, hence it neglects what happens with the portfolio value along the way. Moreover, VaR assumes the current positions being fixed over the holding period. In practice, however, positions are changed almost continuously. It is fair to say, however, that these weaknesses are not peculiar to VaR; other one-period risk measures have the same shortcomings. More serious though is the fact that VaR does *not* measure the potential size of a loss given that the loss exceeds VaR. It is mainly for this reason why VaR is not being used in the Swiss Solvency Test for the determination of the so-called target capital. There, the regulatory capital requirement asks for sufficient capital to be left (on average) in a situation of financial distress in order to ensure a smooth run-off of the portfolio.

The main criticism of VaR, however, is that in general it lacks the property of subadditivity. Care has to be taken when risks are extremely skewed or heavy-tailed, or in case they encounter a special dependency structure. In such circumstances, VaR may not be sub-additive, as the following example with two very heavy-tailed risks shows. The implications for the modeling of economic capital are severe as the concept of diversification breaks down. We come back to this issue later in Sections 5 and 7 when we talk about operational risk losses and their aggregation.

Example 1. Let X_1 and X_2 be two independent random variables with common distribution function $F_X(x) = 1 - 1/\sqrt{x}$ for $x \geq 1$. Observe that the risks X_1, X_2 have infinite mean, and thus are very heavy-tailed. Furthermore, one easily shows that $\mathrm{VaR}_\alpha(X) = (1-\alpha)^{-2}$. Straightforward calculation then yields

$$\begin{aligned} F_{X_1+X_2}(x) &= \mathbb{P}[X_1 + X_2 \leq x] \\ &= \int_1^{x-1} F_X(x-y)\, dF_X(y) \\ &= 1 - 2\sqrt{x-1}/x \\ &< 1 - \sqrt{2/x} \\ &= F_{2X}(x), \end{aligned}$$

where $F_{2X}(u) = \mathbb{P}[2X_1 \leq u]$ for $u \geq 2$. From this, we then conclude that $\mathrm{VaR}_\alpha(X_1 + X_2) > \mathrm{VaR}_\alpha(2X_1)$. Since $\mathrm{VaR}_\alpha(2X_1) = \mathrm{VaR}_\alpha(X_1) + \mathrm{VaR}_\alpha(X_1)$, it follows that

$$\mathrm{VaR}_\alpha(X_1 + X_2) > \mathrm{VaR}_\alpha(X_1) + \mathrm{VaR}_\alpha(X_2),$$

hence demonstrating that VaR is not sub-additive in this case. Note that a change in the risk measure from VaR to expected shortfall (see Definition 2 below), say, is no reasonable way out in this case. The problem being that expected shortfall is infinite in an infinite mean model.

We conclude this section by showing that VaR is sub-additive for normally distributed risks. In fact, one can show that VaR is sub-additive for the wider class of linear combinations of the components of a multivariate elliptical distribution, see for instance McNeil et al. [MFE05], Theorem 6.8.

Example 2. Let X_1, X_2 be jointly normally distributed with mean vector $\boldsymbol{\mu} = (\mu_1, \mu_2)'$ and covariance matrix

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix},$$

where $-1 \leq \rho \leq 1$ and $\sigma_i > 0$, $i = 1, 2$. Let $0.5 \leq \alpha < 1$, then

$$\mathrm{VaR}_\alpha(X_1 + X_2) \leq \mathrm{VaR}_\alpha(X_1) + \mathrm{VaR}_\alpha(X_2). \tag{1}$$

The main observation here is that, since (X_1, X_2) is bivariate normally distributed, X_1, X_2 and $X_1 + X_2$ all have univariate normal distributions. Hence it follows that

$$\mathrm{VaR}_\alpha(X_i) = \mu_i + \sigma_i\, q_\alpha(N), \qquad i = 1, 2,$$
$$\mathrm{VaR}_\alpha(X_1 + X_2) = \mu_1 + \mu_2 + \sqrt{\sigma_1^2 + 2\rho\sigma_1\sigma_2 + \sigma_2^2}\; q_\alpha(N).$$

Here $q_\alpha(N)$ denotes the α-quantile of a standard normally distributed random variable. The assertion in (1) now follows because of $q_\alpha(N) \geq 0$ (since $0.5 \leq \alpha < 1$) and $(\sigma_1^2 + 2\rho\sigma_1\sigma_2 + \sigma_2^2)^{1/2} \leq \sigma_1 + \sigma_2$ (since $\rho \leq 1$).

Expected shortfall

As mentioned earlier, for a given level α, VaR_α does not give information on the loss sizes beyond this quantile. To circumvent this problem, Artzner et al. [ADEH99] considered the notion of expected shortfall or conditional tail expectation instead.

Definition 2. *Let X be a risk and $\alpha \in (0, 1)$. The expected shortfall or conditional tail expectation is defined as the conditional expected loss given that the loss exceeds $\mathrm{VaR}_\alpha(X)$:*

$$\mathrm{ES}_\alpha(X) = \mathbb{E}[X | X > \mathrm{VaR}_\alpha(X)]\,.$$

Intuitively, $\mathrm{ES}_\alpha(X)$ represents the average loss in the worst $100(1-\alpha)\%$ cases. This representation is made more precise by observing that for a continuous random variable X one has

$$\mathrm{ES}_\alpha(X) = \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}_\xi(X)\,d\xi, \qquad 0 < \alpha < 1.$$

For continuous risks X expected shortfall, as defined in Definition 2, is a coherent risk measure, see Artzner et al. [ADEH99]. For risks which are not continuous, a slight modification of Definition 2 leads to a coherent, i.e. subadditive risk measure; see McNeil et al. [MFE05], Section 2.2.4.

2.2 Risk factor mapping and loss portfolios

Denote the value of a portfolio at time t by $V(t)$. The loss of the portfolio over the period $[t, t+h]$ is given by

$$L_{[t,t+h]} = -\big(V(t+h) - V(t)\big).$$

Note our convention to quote losses as positive values. Following standard risk management practice, the portfolio value is modeled as a function of a d-dimensional random vector $\mathbf{Z}(t) = (Z_1(t), Z_2(t), \ldots, Z_d(t))'$ of risk factors Z_i. Hence,

$$V(t) = V(t; \mathbf{Z}(t)). \tag{2}$$

The representation (2) is known as mapping of risks. Representing a financial institution's portfolio as a function of underlying market-risky instruments constitutes a crucial step in any reasonable risk management system. Indeed, any potential risk factor *not* included in this mapping will leave a blind spot on the resulting risk map.

It is convenient to introduce the vector $\mathbf{X}_{[t,t+h]} = \mathbf{Z}(t+h) - \mathbf{Z}(t)$ of risk factor changes for the portfolio loss $L_{[t,t+h]}$. It can be approximated by $L^{\Delta}_{[t,t+h]}$, where

$$L^{\Delta}_{[t,t+h]} = (\nabla V)' \mathbf{X}_{[t,t+h]} \tag{3}$$

provided the function $V : \mathbb{R}^d \to \mathbb{R}$ is differentiable. Here, ∇f denotes the vector of partial derivatives $\nabla f = (\partial f/\partial z_1, \ldots, \partial f/\partial z_d)'$. Observe that in (3) we suppressed the explicit time dependency of V. The approximation (3) is convenient as it allows one to represent the portfolio loss as a linear function of the risk factor changes, see also Section 4.1. The linearity assumption can be viewed as a first-order approximation (Taylor series expansion of order one) of the risk factor mapping. Obviously, the smaller the risk factor changes, the better the quality of the approximation.

3 Credit risk

Credit risk is the risk of default or change in the credit quality of issuers of securities to whom a company has an exposure. More precisely, default risk is the risk of loss due to a counter-party defaulting on a contract. Traditionally, this applies to bonds where debt holders are concerned that the counter-party might default. Rating migration risk is the risk resulting from changes in future default probabilities. For the modeling of credit risk, the following elements are therefore crucial:

- *default probabilities*: probability that the debtor will default on its obligations to repay its debt;
- *recovery rate*: proportion of the debt's par value that the creditor would receive on a defaulted credit, and
- *transition probabilities*: probability of moving from one credit quality to another within a given time horizon.

In essence, there are two main approaches for the modeling of credit risk, so-called *structural models* and *reduced form* or *intensity based* methods.

3.1 Structural models

Merton [Mer74] proposed a simple capital structure of a firm where the dynamics of the assets are governed by a geometric Brownian motion:

$$dA(t) = A(t)(\mu\, dt + \sigma\, dW(t)), \qquad t \in [0, T].$$

In its simplest form, an obligor's default in a structural model is said to occur if the obligor's asset value $A(T)$ at time T is below a pre-specified deterministic barrier x, say. The default probability can then be calculated explicitly:

$$\mathbb{P}[A(T) \le x] = \Phi\left(\frac{\log(x/A(0)) - (\mu - \sigma^2/2)T}{\sigma\sqrt{T}}\right).$$

Here Φ denotes the cumulative distribution function of a standard normal random variable, i.e. $\Phi(x) = 1/\sqrt{2\pi}\int_{-\infty}^{x} \exp\{-y^2/2\}\, dy$. Various extensions of Merton's original firm value model exist. For instance, one can let the barrier x be a (random) function of time.

3.2 Reduced form models

In a reduced form pricing framework, it is assumed that the default time τ is governed by a risk neutral default intensity process $\lambda = \{\lambda(t) : t \ge 0\}$. That is, default is defined as the first arrival time (jump time) τ of a counting process with intensity λ. It can be shown that the conditional probability at time t, given all information at that time, of survival to a future time T, is given by

$$p(t,T) = \mathbb{E}_{\mathbb{Q}}\big[e^{-\int_t^T \lambda(u)\,du}|\mathcal{F}_t\big] \tag{4}$$

From (4) one immediately recognizes the analogy between an intensity process λ and a short interest rate process r for the time-t price of a (default free) zero-coupon bond maturing at time T. The latter is given by

$$P(t,T) = \mathbb{E}_{\mathbb{Q}}\big[B(t)/B(T)|\mathcal{F}_t\big] = \mathbb{E}_{\mathbb{Q}}\big[e^{-\int_t^T r(u)\,du}|\mathcal{F}_t\big] ,$$

where $B(t) = \exp\{\int_0^t r(s)\,ds\}$ denotes the risk free bank account numéraire. As shown by Lando [Lan98], the *defaultable* bond price at time t (assuming zero recovery) is then given by

$$\bar{P}(t,T) = \mathbb{E}_{\mathbb{Q}}\big[e^{-\int_t^T (r(u)+\lambda(u))\,du}|\mathcal{F}_t\big] ,$$

provided default has not already occurred by time t.

Reduced form models can be extended to allow for non-zero recovery. Duffie and Singleton [DS99] for instance introduce the concept of recovery of market value (RMV), where recovery is expressed as a fraction of the market value of the security just prior to default. Formally, it is assumed that the claim pays $(1-L(t))V(t-)$, where $V(t-) = \lim_{s\uparrow t} V(s)$ is the price of the claim just before default, and $L(t)$ is the random variable describing the fractional loss of market value of the claim at default. Under technical conditions, Duffie and Singleton [DS99] show that

$$\bar{P}(t,T) = \mathbb{E}_{\mathbb{Q}}\big[e^{-\int_t^T (r(u)+\lambda(u)L(u))\,du}|\mathcal{F}_t\big] .$$

Here, $r(u) + \lambda(u)L(u)$ is the default-adjusted short rate.

3.3 Credit risk for regulatory reporting

Compared to the original 1988 Basel accord and its amendments, Basel II better reflects the relative credit qualities of obligors based on their credit ratings. Two approaches are being proposed under Basel II, namely

(A) Standardized approach
(B) Internal-ratings-based approach

The *standardized approach* better recognizes the benefits of credit risk migration and also allows for a wider range of acceptable collateral. Under the *internal-ratings-based approach*, a bank can — subject to the bank's regulator approval — use its own internal credit ratings. The ratings must correspond to the one-year default probabilities and have to be in place for a minimum of three years.

The assessment of credit risk under the Solvency II regime essentially follows the Basel II principles. Within the Swiss Solvency Test for instance, the Basel II standardized approach is being advocated. Portfolio models are acceptable too, provided they capture the credit migration risk. It is for this reason why the CreditRisk+ model for instance would not be permissible within the Swiss Solvency Test as this model only covers the default risk, but not the credit migration risk.

4 Market risk

Market risk is the risk that the value of an investment will decrease due to moves in market risk factors. Standard market risk factors are interest rates, stock indices, commodity prices, foreign exchange rats, real estate indices, etc.

4.1 Market risk models

Variance-covariance

The standard analytical approach to estimate VaR or expected shortfall is known as *variance-covariance method*. This means that the risk factor changes are assumed to be samples from a multivariate normal distribution, and that the loss is represented as a linear function of the risk factor changes, see Section 2.2 for more details. This approach offers an analytical solution, and it is much faster to calculate the risk measures in a parametric regime than performing a simulation. However, a parametric approach has significant limitations. The assumption of normally distributed risk factor changes may heavily underestimate the severeness of the loss distribution. Moreover, linearization may be a poor approximation of the risk factor mapping.

Historical simulation

The second well-established approach for measuring market risk exposure is the *historical simulation method*. Instead of estimating the loss distribution under some explicit parametric model for the risk factor changes, one uses the empirical distribution of the historical loss data. VaR and expected shortfall can then either be estimated directly from the simulated data or by first fitting a univariate distribution to the loss data. The main advantage of this approach is its simplicity of implementation. No statistical estimation of the distribution of the risk factor changes is required. In particular, no assumption on the interdependence of the underlying risk factors is made. On the downside, it may be difficult to collect enough historical data of good quality. Also, the observation period is typically not long enough such that samples of extreme changes in the portfolio value cannot be found. Therefore, adding suitable stress scenarios is very important.

Monte Carlo simulation

The idea behind the *Monte Carlo method* is to estimate the loss distribution under some explicit parametric model for the risk factor changes. To be more precise, one first fits a statistical model to the risk factor changes. Typically, this model is inferred from the observed historical data. Monte Carlo-simulated risk factor changes then allow one to make inferences about the loss distribution and the associated risk measure. This approach is very general, albeit it may require extensive simulation.

4.2 Conditional versus unconditional modeling

In an *unconditional approach*, one neglects the evolution of risk factor changes up to the present time. Consequently, tomorrow's risk factor changes are assumed to have the same distribution as yesterday's, and thus the same variance as experienced historically. Such a stationary model corresponds to a long-term view and is often appropriate for insurance risk management purposes. However, empirical analysis often reveals that the volatility σ_t of market risk factor changes $X(t)$, *conditionally* on their past, fluctuates over time. Sometimes, the market is relatively calm, then a crisis happens, and the volatility will suddenly increase. Time series models such as GARCH type models allow the variance σ_{t+1}^2 to vary through time. They are suited for a short-term perspective. GARCH stands for generalized autoregressive conditional heteroskedasticity, which in essence means that the conditional variance on one day is a function of the conditional variances on the previous days.

4.3 Scaling of market risks

A risk measure's holding period should be related to the liquidity of the assets. If a financial institution runs into difficulties, the holding period should cover the time necessary to raise additional funds for corrective actions. The Basel II VaR approach for market risk for instance requires a holding period of ten days (and a confidence level $\alpha = 0.99$). The time horizon thus often spans several days, and sometimes even extends to a whole year. While the measurement of short-term financial risks is well established and documented in the financial literature, much less has been done in the realm of long-term risk measurement. The main problem is that long-dated historical data in general is not representative for today's situation and therefore should not be used to make forecasts about the future changes in market risk factors. So the risk manager is typically left with little reliable data to make inference of long term risk measures.

One possibility to close this gap is to *scale* a short-term risk estimate to a longer one. The simplest way to do this is to use the square-root-of-time scaling rule, where a k-day Value at Risk $\mathrm{VaR}^{(k)}$ is scaled with $\sqrt{n}$ in order to get an estimate for the nk-day Value at Risk $\mathrm{VaR}^{(nk)} \approx \sqrt{n}\mathrm{VaR}^{(k)}$. This rule is motivated by the fact that one often considers the *logarithm* of tradable securities, say S, as risk factors. The return over a 10-day period for example is then expressible as $R_{[0,10]} := \log(S(10))/S(0)) = X(1)+X(2)+\cdots+X(10)$, where $X(k) = \log(S(k))-\log(S(k-1)) = \log(S(k)/S(k-1))$, $k = 1, 2, \ldots, 10$. Observe that for independent random variables $X(i)$ the standard deviation of $R_{[0,10]}$ equals $\sqrt{10}$ times the standard deviation of $X(1)$. In this section we analyze under which conditions such scaling is appropriate. We concentrate on unconditional risk estimates and on VaR as risk measure. Recall our convention to quote losses as positive values. Thus the random variable

$L(t)$ will subsequently denote the negative value of one-day log-returns, i.e. $L(t) = -\log(S(t)/S(t-1))$ for some security S.

Scaling under normality

Under the assumption of independent and identically zero-mean normally distributed losses $L(t) \sim \mathcal{N}(0, \sigma^2)$ it follows that the n-day losses are also normally distributed, that is $\sum_{t=1}^{n} L(t) \sim \mathcal{N}(0, n\sigma^2)$. Recall that for a $\mathcal{N}(0, \widetilde{\sigma}^2)$-distributed loss L, VaR is given by $\mathrm{VaR}_\alpha(L) = \widetilde{\sigma}\, q_\alpha(N)$, where $q_\alpha(N)$ denotes the α-quantile of a standard normally distributed variate. Hence the square-root-of-time scaling rule

$$\mathrm{VaR}^{(n)} = \sqrt{n}\, \mathrm{VaR}^{(1)}$$

works perfectly in this case.

Now let a constant value μ be added to the one-day returns, i.e. μ is subtracted from the one-day loss: $L(t) \sim \mathcal{N}(-\mu, \sigma^2)$. Assuming independence among the one-day losses, the n-day losses are again normally distributed with mean value $-n\mu$ and variance $n\sigma^2$, hence $\sum_{t=1}^{n} L(t) \sim \mathcal{N}(-n\mu, n\sigma^2)$. The VaR in this case will be increased by the trend of L. This follows from $\mathrm{VaR}^{(n)} + n\mu = \sqrt{n}\, (\mathrm{VaR}^{(1)} + \mu)$, or equivalently

$$\mathrm{VaR}^{(n)} = \sqrt{n}\, \mathrm{VaR}^{(1)} - (n - \sqrt{n})\mu.$$

Accounting for trends is important and therefore trends should never be neglected in a financial model. Note that the effect increases linearly with the length n of the time period.

To simplify matters, all the models presented below are restricted to the zero-mean case. They can easily be generalized to non-zero mean models, implying that the term $(n - \sqrt{n})\mu$ must be taken into account when estimating and scaling VaR.

Autoregressive models

Next, we consider a stationary autoregressive model of the form

$$L(t) = \lambda L(t-1) + \varepsilon_t,$$

where $(\varepsilon_t)_{t \in \mathbb{N}}$ is a sequence of iid zero-mean normal random variables with variance σ^2 and $\lambda \in (-1, 1)$. Not only are the one-day losses normally distributed, but also the n-day losses:

$$L(t) \sim \mathcal{N}\big(0, \frac{\sigma^2}{1-\lambda^2}\big), \qquad \sum_{t=1}^{n} L(t) \sim \mathcal{N}\big(0, \frac{\sigma^2}{(1-\lambda)^2}(n - 2\lambda \frac{1-\lambda^n}{1-\lambda^2})\big).$$

Hence, making use of $\mathrm{VaR}_\alpha(L) = \widetilde{\sigma}\, q_\alpha(N)$, one obtains

$$\mathrm{VaR}^{(n)} = \sqrt{\frac{1+\lambda}{1-\lambda}\left(n - 2\lambda\frac{1-\lambda^n}{1-\lambda^2}\right)}\,\mathrm{VaR}^{(1)}. \tag{5}$$

Since the square-root expression in (5) tends to $\sqrt{n}$ as $\lambda \to 0$, one concludes that the scaled one-day value $\sqrt{n}\,\mathrm{VaR}^{(1)}$ is a good approximation of $\mathrm{VaR}^{(n)}$ for small values of λ.

For more general models, such as stochastic volatility models with jumps or AR(1)-GARCH(1,1) models, the correct scaling from a short to a longer time horizon depends on the confidence level α and cannot be calculated analytically. In many practical applications, the confidence level varies from 0.95 to 0.99, say. Empirical studies show that for such values of α, scaling a short-term VaR with the square-root-of-time yields a good approximation of a longer-term VaR, see Kaufmann [Kau05]. For smaller values of α, however, the scaled risks tend to *overestimate* the true risks, whereas larger values of α tend to *underestimate* the risks. In the limit $\alpha \to 1$, one should abstain from scaling risks, see Brummelhuis and Guégan [BG00a, BG00b].

Sometimes risk managers are also confronted with the problem of transforming a 1-day VaR at the confidence level $\alpha = 0.95$ to a 10-day VaR at the 0.99 level. From a statistical viewpoint, such scaling should be avoided. Our recommendation is to first try to arrive at an estimate of the 1-day VaR at the 0.99 level and then to make inference of the 10-day VaR by means of scaling.

In this section we analyzed the scaling properties in a VaR context. As a matter of fact, these properties in general do not carry over when replacing VaR through other risk measures such as expected shortfall. In an expected shortfall regime coupled with heavy-tailed risks, scaling turns out to be delicate. For light-tailed risk though the square-root-of-time rule still provides good results when expected shortfall is being used.

5 Operational risk

According to the capital adequacy frameworks as set out by the Basel Committee, the general requirement for banks is to hold total capital equivalent to at least 8% of their risk-weighted assets. This definition was retained of the old capital adequacy framework (Basel I). In developing the revised framework now known as Basel II the idea was to arrive at significantly more risk-sensitive capital requirements. A key innovation in this regard was that operational risk –besides market and credit risk– must be included in the calculation of the total minimum capital requirements. Following the Committee's wording, we understand by operational risk "the risk of losses resulting from inadequate or failed internal processes, people and systems, or external events".

The Basel II framework provides a range of options for the determination of an operational risk capital charge. The proposed methods allow banks and supervisors to select approaches that are most appropriate for a bank's operations. These methods are:

(1) basic indicator approach,
(2) standardized approach,
(3) advanced measurement approach.

Both the *basic indicator approach* as well as the *standardized approach* are essentially volume-based measurement methods. The proxy in both cases is the average gross income over the past three years. These measurement methods are primarily destined for small and medium-sized banks whose exposure to operational risk losses is deemed to be moderate. Large internationally active banks, on the other hand, are expected to implement over time a more sophisticated measurement approach. Those banks must demonstrate that their approach is able to capture "severe tail loss events". More formally, banks should set aside a capital charge C_{Op} for operational risk in line with the 99.9% confidence level on a one-year holding period. Using VaR as risk measure, this approach is known as *loss distribution approach* (LDA). It is suggested to use $\sum_{k=1}^{8} \mathrm{VaR}_\alpha(L_k)$ for a capital charge and to allow for a capital reduction through diversification under appropriate dependency assumptions. Here, L_k denotes the one-year operational risk loss of business line k. The choice of 8 business lines and their precise definition is to be found in the Basel II Accord, banks are allowed to use fewer or more.

It is at this point where one has to sway a warning flag. A recent study conducted by Moscadelli [Mos04] reveals that operational loss amounts are *very* heavy-tailed. This stylized fact has been known before, albeit not in such an unprecedented way. Moscadelli's analysis suggests that the loss data from six out of eight business lines come from an infinite mean model! An immediate consequence is that standard correlation coefficients between two such one-year losses do not exist. Nešlehová et al. [NEC06] in their essay carry on with the study of Moscadelli's data and show the serious implications extreme heavy-tailedness can have on the economic capital modeling, in particular when using VaR as a risk measure. Note that it is not the determination of a VaR per se that causes problems in an infinite mean model. Rather, it is the idea of capital reduction due to aggregation or pooling of risks that breaks down in this case, see Example 1 on page 5. We will come back to this issue later in Section 7.

Operational risk is also part of Solvency II and most of the insurance industry's national risk-based standard models. In the realm of the Swiss Solvency Test for instance it suffices to assess operational risk on a pure qualitative basis. Other models, such as the German GDV model for instance, require a capital charge for operational risk. This charge is mainly volume-based, similar to the Basel II basic indicator or standardized approach.

6 Insurance risk

6.1 Life insurance risk

Life insurance contracts are typically characterized by long-term financial promises and guarantees towards the policyholders. The actuary's main task has therefore been to forecast the future liabilities, that is to set up sufficient reserves in order that the company can meet its future obligations. Ideally, the modern actuary should also be able to form an opinion on how many assets will be required to meet the obligations and on how the asset allocation should look like from a so-called *asset and liability management* (ALM) perspective. So life insurance companies are heavily exposed to *reserve* risk. Under reserve risk, we understand the risk that the actual claims experience deviates from the booked reserves. Booked reserves are always based on some accounting conventions and are determined in such a way that sufficient provisions are held to cover the *expected* actuarial liabilities based on the tariffs. Typically, these reserves are formula-based, that is, a specific calculation applied individually to each contract in force, then summed up, yields the reserves. Even though they include a margin for prudence, the reserves may prove insufficient in the course of time because of e.g. demographic changes.

Reserve risk can further be decomposed into the following sub-categories:

(A) stochastic risk,
(B) parametric risk,
(C) model risk.

The *stochastic risk* is due to the variation and severity of the claims. In principle, the stochastic risk can be diversified through a greater portfolio and an appropriate reinsurance program. By ceding large individual risks to a reinsurer via a surplus share for instance, the portfolio becomes aptly homogeneous.

Parametric risk arises from the fact that tariffs can be subject to material changes over time. For example, an unprecedented increase in longevity implies that people will draw annuities over a longer period. It is the responsibility of the (chief) actuary to continually assess and monitor the adequacy of the reserves. Periodic updates of experience data give insight into the adequacy of the reserves.

By *model risk* finally we understand the risk that a life insurance company has unsuitable reserve models in place. This can easily be the case when life insurance products encompass a variety of policyholder options such as e.g. early surrender or annuity take-up options. Changing economic variables and/or an increase in the longevity can result in significant future liabilities, even when the options were far out of the money at policy inception. It was a combination of falling long-term interest rates and booming stock markets coupled with an increase in longevity that put the solvency of Equitable Life, a UK insurer, at stake and led to the closure of new business. The reason for this was that so-called guaranteed annuity options dramatically increased in value and subsequently constituted a significant liability which was neither priced nor reserved for. Traditional actuarial pricing and reserving methods based on the expectation pricing principle prove useless in this context were it not for those policyholders who behave in a financially irrational way. Indeed, empirical studies may reveal that there is *no* statistical evidence supporting a link between the surrender behavior and the level of market interest rates. Arbitrage pricing techniques are always based on the assumption of financially rational policyholder behavior though. This means that a person would surrender its endowment policy at the first instant when the actual payoff exceeded the value of continuation.

The merits of arbitrage pricing techniques are that they provide insight into the mechanism of embedded options, and consequently these findings should be used when designing new products. This will leave an insurance company immune against potential future changes in the policyholders' behavior towards a more rational one from a mathematical economics point of view.

6.2 Modeling parametric life insurance risk

In the following, we will present a model that allows for the quantification of parametric life insurance risk. This model is being used within the Swiss Solvency Test. In essence, it is a variance-covariance type model, that is

- risk factor changes have a multivariate normal distribution, and
- changes in the best estimate value of liabilities linearly depend on the risk factor changes.

More formally, it is assumed that for risk factor changes $\mathbf{X}$ and weights b

$$\Delta L = \mathbf{b}'\mathbf{X}$$

where $L = L(\mathbf{Z}(t))$ denotes the best estimate value of liabilities at the valuation date t, and $\mathbf{Z}(t) = \big(Z_1(t), Z_2(t), \ldots, Z_d(t)\big)'$ is the vector of (underwriting) risk factors. Best estimate values are unbiased (neither optimistic, nor pessimistic, nor conservative) estimates which employ the most recent and accurate actuarial and financial market information. Best estimate values are without any (safety) margins whatsoever. Typically, the value L is obtained by means of projecting the future cash flows and subsequent discounting with the current risk-free yield curve.

Again, we denote the risk factor changes by $\mathbf{X}(t) = \mathbf{Z}(t) - \mathbf{Z}(t-1)$. Within the Swiss Solvency Test, the following set of risk factors is considered:
The risk factor "mortality" for example refers to the best estimate one-year mortality rates q_x, q_y respectively (second-order mortality rates). The risk factor "longevity" refers to the improvement of mortality which is commonly expressed in exponential form

Table 1. Life insurance risk factors in the Swiss Solvency Test.

(R1)	mortality	(R4)	recovery
(R2)	longevity	(R5)	surrender/lapse
(R3)	disability	(R6)	annuity take-up

$$q(x,t) = q(x,t_0)e^{-\lambda_x(t-t_0)}\,, \qquad t \geq t_0,$$

where $q(x,t_0)$ stands for the best estimate mortality rate of an x year old male at time t_0.

Typically, no analytical solutions exist for the partial derivatives $b_k = \partial L/\partial z_k$, and hence they have to be approximated numerically by means of sensitivity calculations:

$$b_k \approx \frac{L(\mathbf{Z} + \varepsilon\mathbf{e}_k) - L(\mathbf{Z})}{\varepsilon}$$

for ε small, e.g. $\varepsilon = 0.1$. Here, $\mathbf{e}_k = (0,\ldots,0,1,0,\ldots,0)'$ denotes the kth basis vector in $\mathbb{R}^d$. Combining everything, one concludes that the change ΔL has a univariate normal distribution with variance $\mathbf{b}'\Sigma\mathbf{b}$, i.e. $\Delta L \sim \mathcal{N}(0, \mathbf{b}'\Sigma\mathbf{b})$. Here it is assumed that the dependence structure of the underwriting risk factor changes are governed by the covariance matrix Σ. Note that Σ can be decomposed into its correlation matrix R and a diagonal matrix Δ comprising the risk factor changes' standard deviations on the diagonal. Hence, $\Sigma = \Delta R \Delta$. Both the correlation coefficients and the standard deviations are based on expert opinion; no historical time series exists from which estimates could be inferred. Knowing the distribution of ΔL, one can apply a risk measure ϱ to arrive at a capital charge for the parametric insurance risk. Within the Swiss Solvency Test, one uses expected shortfall at the confidence level $\alpha = 0.99$.

Table 2 shows the correlation matrix R currently being used in the Swiss Solvency Test, whereas Table 3 contains the standard deviations of the underwriting risk factor changes.

6.3 Non-life insurance risk

For the purpose of this article, the risk categories (in their general form) already discussed for life above, also apply. Clearly there are many distinctions at the product level. For instance, in non-life we often have contracts over a shorter time period, frequency risk may play a bigger role (think for instance of hail storms) and especially in the realm of catastrophe risk, numerous specific methods have been developed by non-life actuaries. Often techniques borrowed from (non-life) risk theory are taken over by the banking world. Examples are the modeling of loss distributions, the axiomatization of risk measures, IBNR and related techniques, Panjer recursion, etc. McNeil et al. [MFE05] yield an

Table 2. Correlation matrix R of the life insurance risk factor changes.

		Individual						Group					
		R1	R2	R3	R4	R5	R6	R1	R2	R3	R4	R5	R6
Individual	R1	1	0	0	0	0	0	1	0	0	0	0	0
	R2	0	1	0	0	0	0	0	1	0	0	0	0
	R3	0	0	1	0	0	0	0	0	1	0	0	0
	R4	0	0	0	1	0	0	0	0	0	1	0	0
	R5	0	0	0	0	1	0.75	0	0	0	0	1	0.75
	R6	0	0	0	0	0.75	1	0	0	0	0	0.75	1
Group	R1	1	0	0	0	0	0	1	0	0	0	0	0
	R2	0	1	0	0	0	0	0	1	0	0	0	0
	R3	0	0	1	0	0	0	0	0	1	0	0	0
	R4	0	0	0	1	0	0	0	0	0	1	0	0
	R5	0	0	0	0	1	0.75	0	0	0	0	1	0.75
	R6	0	0	0	0	0.75	1	0	0	0	0	0.75	1

Table 3. Standard deviations of the life insurance risk factor changes (in percentage).

	Individual							Group						
	R1	R2	R3	R4	R5	R6	R7	R1	R2	R3	R4	R5	R6	R7
σ_i	5	10	10	10	25	0	10	5	10	20	10	25	0	10

exhaustive overview on the latter techniques and refer to them as Insurance Analytics. For a comprehensive summary of the modeling of loss distributions, see for instance Klugman et al. [KPW04]. An example stressing the interplay between financial and insurance risk is Schmock [Sch99].

In the Swiss Solvency Test non-life model the aim is to determine the change in risk bearing capital within one year due to the variability of the technical result. The model is based on the accident year principle. That is, claims are grouped according to the date of occurrence (and not according to the date or year when they are reported). Denoting by $[T_0, T_1]$ with $T_1 = T_0 + 1$ the one-year time interval under consideration, the technical result within $[T_0, T_1]$ is not only determined by the claims occurring in this period, but also by the claims that have previously occurred and whose settlement stretches across $[T_0, T_1]$.

The current year claims are further grouped into high frequency-small severity claims ("small claims") and low frequency-high severity claims ("large claims"). It is stipulated that the total of small claims has a gamma distribution, whereas in the large claims regime a compound Poisson distribution with Pareto distributed claim sizes is used.

As for the claims that have occurred in the past and are not yet settled, the focus is on the annual reserving result; it is defined as the difference between the sum of the claim payments during $[T_0, T_1]$ plus the remaining provisions after T_1 minus the provisions that were originally set up at time T_0. Within the Swiss Solvency Test, this one-year reserve risk is modeled by means of a (shifted) log-normally distributed random variable.

To obtain the ultimate probability distribution of the non-life risk, one first aggregates the small claims and the large claims risk, thereby assuming independence between these two risk categories. A second convolution is then required to combine the resulting current year risk with the reserve risk, again assuming independence.

7 Aggregation of risks

A key issue for the economic capital modeling is the aggregation of risks. Economic capital models are too often based on the tacit assumption that risk can be diversified via aggregation. For VaR in the context of very heavy-tailed distributions, however, the idea of a capital relief due to pooling of risks may shipwreck, see Example 1 on page 5 where it is shown that VaR is not sub-additive for an infinite mean model. The (non-) existence of subadditivity is closely related to Kolmogorov's strong law of large numbers, see Nešlehová et al. [NEC06].
In a formal way, diversification could be defined as follows:

Definition 3. *Let $X_1, \dots, X_n$ be a sequence of risks and ϱ a risk measure. Diversification is then expressed as*

$$\mathcal{D}_\varrho := \sum_{k=1}^{n} \varrho(X_k) - \varrho\big(\sum_{k=1}^{n} X_k\big).$$

Extreme heavy-tailedness is one reason why VaR fails to be sub-additive. Another reason overthrowing the idea of diversification is extreme skewness of risks as the following simple example demonstrates. Assume that a loss of EUR 10 million or more is incurred with a probability of 3% and that the loss will be EUR 100'000 with a probability of 97%. In this case the VaR at the 95% level is EUR 100'000, while aggregating two such independent losses yields a VaR of more than EUR 10 million.

The modeling of dependence is a central element in quantitative risk management. In most cases, the assumption of *independent* (market-) risky instruments governing the portfolio value is too simplistic and unrealistic. Correlation is by far the most used technique in modern finance and insurance to describe dependence between risks. And yet correlation is only one particular measure of stochastic dependence among others. Whereas correlation is perfectly suited for elliptically distributed risks, dangers lurk if correlation is used in a non-elliptical world. Recall that independence of two random variables always implies their uncorrelatedness. The converse, however, does in general not hold.

We have shown in Example 2 on page 6 that VaR is sub-additive in a normal risks regime. Indeed, this fact can be used to aggregate market and insurance risk in a variance-covariance type model, see Sections 4.1 and 6.2. There, the required economic capital when combining market and insurance risks will naturally be reduced compared to the stand-alone capital requirements.

The above example with extremely skewed risks also shows that independence can be worse than comonotonicity. Comonotonicity means that the risks $X_1, \ldots, X_d$ are expressible as increasing functions of a single random variable, Z say. In the case of comonotonic risks VaR is additive, see for instance McNeil et al. [MFE05], Proposition 6.15. For given marginal distribution functions and unknown dependence structure, it is in fact possible to calculate upper and lower bounds for VaR, see Embrechts et al. [EHJ03]. However, these bounds often prove inappropriate in many practical risk management applications. As a consequence, the dependence structure among the risks needs to be modeled explicitly – if necessary by making the appropriate assumptions.

8 Summary

In this paper, we have summarized some of the issues underlying the quantitative modeling of risks in insurance and finance. The taxonomy of risk discussed is of course incomplete and very much driven by the current supervisory process within the financial and insurance services industry. We have hardly vouched upon the huge world of risk mitigation via financial derivatives and alternative risk transfer, like for instance catastrophe bonds. Nor did we discuss in any detail specific risk classes like liquidity risk and model risk; for the latter, Gibson [Gib00] yields an introduction. Beyond the discussion of quantitative risk measurement and management, there is also an increasing awareness that qualitative aspects of risk need to be taken seriously. Especially through the recent discussions around operational risk, this qualitative aspect of risk management became more important. Though modern financial and actuarial techniques have highly influenced the quantitative modeling of risk, there is also a growing awareness that there is an end to the line for this quantitative approach. Though measures like VaR and the whole statistical technology behind it have no doubt had a considerable influence on the handling of modern financial instruments, hardly anybody might believe that a single number like VaR can really summarize the overall complexity of risk in an adequate way. For operational risk, this issue is discussed in Nešlehová et al. [NEC06]; see also Klüppelberg and Rootzén [KR99].

Modern risk management is being applied to areas of industry well beyond the financial ones. Examples include the energy sector and the environment. Geman [Gem05] gives an overview of some of the modeling and risk management issues for these markets. A more futuristic view on the types of risk modern society may want to manage is given in Shiller [Shi03].

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